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Resource allocation in non-Walrasian environments*

Some analytical and simulation results

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We are interested in resource allocation mechanisms that perform satisfactorily in environments which are problematic for the Walrasian equilibrium theory and the tatonnement. The B(idding)-process of Hurwicz–Radner–Reiter is informationally decentralized and has appealing normative properties The speed of convergence of the B-process is characterized in an economy with indivisible commodities. A modification of this process is shown to converge in an economy with both divisible and indivisible goods. Some simulation results on the B-process and its modifications throwing light on its performance in non-Walrasian environments are also reported.

1. Introduction

The problem of developing a logical framework which can be used to analyze, evaluate and perhaps formally compare alternative organizations (or 'mechanisms') for resource allocation has continued to engage the attention of economic theorists [see, e.g., Marschak (1972), Reiter (1986) and Hurwicz (1986)]. Some of the criteria for evaluating the performance of a mechanism have been suggested by the properties of Walrasian models of equilibrium and tatonnement in an economy with many agents. These include, for example, non-wastefulness, unbiasedness, preservation of privacy (informational decentralization), efficiency of communication (size and complexity of messages involved in the functioning of the process) and incentive compatibility.

It is well known that the presence of indivisible commodities leads to

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obvious difficulties in developing the theory of existence and optimality of a Walrasian equilibrium [see, e.g., Koopmans (1957)][1] Thus, when the classical assumptions of convexity or absence of externalities are unrealistic, one is interested in designing 'new allocation mechanisms that would meet our standards of performance (non-wastefulness or unbiasedness) in non-classical environments and still qualify as informationally decentralized'.[2] Yet another motivation for experimenting with alternative mechanisms comes from dynamics: examples were given by Scarf (1960) and Gale (1963) showing that (in the presence of strong income effects) a tatonnement may not converge at all, or that a particular ('fair') equilibrium cannot be attained (is globally unstable). In this paper we present some analytical and simulation results on stochastic B(idding)-processes. Such processes were introduced by Hurwicz–Radner–Reiter (1975) (to be referred to as H–R–R henceforth). A related process (the Random Ascent or RA process) was studied subsequently by Mitsui (1981, 1984). Our interest in the B-process is primarily due to some remarkable properties that hold even when the economy has indivisible goods: it is an informationally decentralized process which is non-wasteful, unbiased, and converges to a Pareto-optimal state with probability one (in finite time when there are only indivisible goods). Thus, when one turns to environments that pose difficulties for the Walrasian model (for brevity, non-Walrasian environments) an examination of the B-process provides a convenient point of departure.

The analytical results on B-processes presented in sections 2–3 supplement the literature in two directions. In section 2 we focus on the question of speed of convergence in an economy consisting exclusively of indivisible goods. Even when one establishes that an adjustment process is stable in some sense, the question of the speed of convergence seems to us to be particularly important. Economic agents have finite life spans, of which only a fraction can be allocated to economic activity. Thus, if agents are not allowed to trade until an equilibrium is reached (in a tatonnement) or a Pareto-improving transaction is found (as in the B-process), the economic significance (from both the descriptive and predictive points of view) of a model of equilibrium and adjustment is somewhat limited if attainment of equilibrium turns out to be painfully slow. The importance of studying the

[1]One approach is to study approximate Walrasian equilibria in environments with non-convex preferences and/or with non-convex consumption sets [see Anderson, Khan and Rashid (1982) and Khan and Rashid (1982)] Thus, for instance, the former paper demonstrates the existence of a price vector which bounds the per capita excess demand by $C/\sqrt{N}$, where N is the number of agents, and C is independent of preferences. While this result helps in gauging the extent of the difficulty faced by the Walrasian mechanism in non-convex environments, it still means that aggregate excess demand can grow by $\sqrt{N}$; furthermore, there still remains the problem of achieving this price through an adjustment process.

[2]See Hurwicz–Radner–Reiter (1975, p. 188).

rate of convergence was recognized by H–R–R (1975, pp. 190–191) although it has received relatively little attention in the stability literature [surveyed, e.g., by Hahn (1982)]. To keep the exposition self-contained, we outline a simplified version of the H–R–R process in an economy consisting exclusively of K indivisible commodities (a detailed exposition of a more general framework is in H–R–R). The main results of section 2 (see Proposition 2.1) indicate that the B-process converges at an exponential rate to a Pareto optimal allocation. More precisely, there exist positive numbers α and C such that for any initial distribution U_0 over the states, one has

$$\text{Probability } [U_t \text{ is not Pareto optimal}] \leqq C\,\mathrm{e}^{-\alpha t}$$

where U_t is the vector of utility levels of the current allocation at the end of t periods. The numbers α and C are derived from the description of the economy.

A second direction in which we have made some analytical progress is a study of the case in which the economy has both divisible and indivisible goods (this 'mixed' case was not covered in H–R–R or Mitsui). Here we propose a modification of the B-process (the 'Surplus Reallocation Process') that is still informationally decentralized and converges to a Pareto-optimal allocation with probability one. The primary difference between our process and, say, the divisible economy of H–R–R, is this: in H–R–R the message communicated by each agent consists of the intersection of a cube of fixed radius with the 'at least as good as' set relative to the current allocation. This 'set-valued' message can, therefore, be quite complex, requiring a detailed description of the local preference pattern. In our case, the agent (as in the indivisible economy) selects a single commodity bundle and communicates the net trade (i.e. a vector with $M+K$ coordinates if there are M divisible goods and K indivisible goods) corresponding to this bid to the referee of the process. The referee then sums up the agents' net trade vectors. If (and only if) the total corresponding to each of the M divisible commodities is zero or negative, and that corresponding to each of the K indivisible goods is zero, are the agents' bids accepted by the referee. Any 'surplus' for some divisible commodity (i.e. a strictly negative total net trade for that commodity) is reallocated by him to all the agents by a simple averaging rule.

The main advantage of transmission of 'point-valued' messages, as is the case in the Surplus Reallocation Process described above, is that it can significantly lower the aggregate costs of communication (if they increase with complexity of messages). Our modification, outlined in section 3, was in fact suggested by our computer-experiments on the sensitivity of the speed of convergence (see the comments in section 4). The importance of studying the mixed economy and of simplifying the structures of bids in the divisible case (while retaining the optimality and convergence properties for non-convex environments) was stressed by H–R–R (1975, see p. 19, 'A future extension of our results to "mixed" cases would be helpful' and also pp. 195–196, 'the construction of the bid in the divisible cases can be quite complex.... It is an open question to what extent one could hope to simplify the structure of bids in the divisible case'). We show (see Theorem 3.1) that the Surplus Reallocation Process described above converges to a Pareto optimal position with probability one in economies with both divisible and indivisible goods. We have not, however, made any formal analytical comparison between our process and the B-process of H–R–R.

It is also worthwhile to indicate the differences between Mitsui's work and ours. Mitsui's process is also a stochastic adjustment process which converges in the limit to a Pareto optimal position. In Mitsui's framework, there is a central authority which generates proposals, which are then submitted to the agents, each of whom decides whether to accept or reject the proposal. In contrast, our process, like the B-process of H–R–R, would seem closer to the spirit of decentralization, since it is the agents themselves who initiate the proposals, while engaged in randomized search for utility improving trades. In addition, Mitsui does not cover the class of environments in which both divisible and indivisible goods are present. On the other hand, because the center is initiating the proposals, the class of economies for which Mitsui's result holds is much larger, and also includes economies with consumption externalities, unlike the case here.

In section 4 we summarize the lessons from computer experiments. The simulations were run on the Cornell Supercomputer (IBM 3090/600E). We report (in section 4.1) on simulations of the B-process and its modification in Scarf's economy. Performance is measured as follows: we fix the total number of bidding periods at 1,000, and compute the allocative efficiency achieved by each process within this time. The simulations reveal that both the B-process and its modification perform fairly well. On a scale where the initial allocation is at a distance of 1.0 from the Pareto Frontier, the B-process and the SR process reached within 0.01 and 0.11 of the Frontier, respectively (averaged over 100 sample paths).

The B-process also does well in Gale's example [see Bala (1989) for details]. In this economy, there are three Walrasian equilibria, with two assigning zero income to each agent. The third 'symmetric' equilibrium is appealing as a 'fair' one, but is globally unstable. As Chipman (1965) remarked, 'both "unfair" solutions are possible competitive equilibria, but the "fair" one is an unstable equilibrium, and therefore, could not be achieved in the framework of the competitive system'. A simulation of (H–R–R) B-process converged on an average of 105 periods, with the mean utility for the agents being close to the utility corresponding to the fair equilibrium. Thus, our simulations indicate that the B-process performs quite satisfactorily in the cases that are so problematic for the tatonnement.[3]

Section 4.2 reports some simulations for the indivisible case. The simulations reveal that the rate of convergence is quite sensitive to the search process employed by the agents. For instance, if on average agents make high bids relative to the total endowments, then the process converges very slowly. Two examples are particularly revealing and ought to be stressed. One shows that in an economy consisting exclusively of indivisible goods, while a B-process converges in finite time with probability one, there may be a long period of 'inaction' or 'no transactions' (i.e., Pareto improving trades may not take place for 99,999 periods!). The other example (in section 4.3) concerns an economy where the dynamic performance of a B-process is adversely affected by incentive incompatibility. Specifically, an agent has an incentive to select a bidding distribution that will enable it to converge to an advantageous Pareto optimal allocation. However, if all the agents attempt this type of manipulation, the expected time taken to converge will tend to infinity.

2. The B-process of H–R–R in an exchange economy with indivisible goods: An outline

We shall discuss the B-process in the context of an *exchange economy.* Assume there are N agents [indexed by $i=1,2,\dots,N$] and K indivisible commodities in the economy. We denote the set of non-negative integers by Z_+. The consumption set of agent i is $C^i \subset Z_+^K$. Each agent also has an endowment vector denoted $\omega^i \in C^i$, and a utility function $u^i : C^i \to R$. In order to highlight the anonymity of the process, the description[4] will proceed in terms of the *net trading sets*

$$Y^i \equiv C^i - \{\omega^i\}.$$

The utility function $\bar{u}^i : Y^i \to R$ is defined on Y^i as $\bar{u}^i(y^i) = u^i(y^i + \omega^i)$ for $y^i \in Y^i$. Hence, we will use the notation u^i for $\bar{u}^i$ even though the domain is different. Each agent also has a *bidding distribution* P^i defined on Y^i. We assume that

[3]As regards alternative adjustment processes for the competitive mechanism, Van der Laan and Talman (1987) have constructed a globally stable algorithm for obtaining competitive equilibria, when excess demand functions are continuously differentiable. Their process converges to the unfair equilibrium in the Gale example. On the other hand, their process does converge to the competitive equilibrium in the Scarf economy. Nevertheless we still choose the Scarf example for comparative purposes, since the tatonnement dynamics remain more compelling from an economic point of view. For the Scarf economy, see also Hirota (1985) for a genericity result.

[4]Some examples will be given in terms of consumption sets.

the *support* of P^i is Y^i, i.e. $P^i(y^i)>0$ for every $y^i \in Y_i$. P^i defines a family of conditional probability distributions $\{P^i(\cdot|y^i), y^i \in Y^i\}$ on Y^i given by

$$P^i(x^i|y^i)=\begin{cases}0 & \text{iff } u^i(x^i)<u^i(y^i) \\ P^i(x^i)/P^i(G^i(y^i)) & \text{otherwise}\end{cases},$$

where $G^i(y^i)$ is the upper contour set, i.e. $G^i(y^i)\equiv\{z^i \in Y^i|u^i(z^i)\geqq u^i(y^i)\}$. Thus, it can be seen that if the agent is currently at y^i, the conditional bidding distribution gives support only to points in the trading set that yield at least as much utility as it has presently. Define the feasible set as:

$$Y_F=\left\{(y^1,\dots,y^N)\in\prod_{i=1}^{N} Y^i \,\middle|\, \sum_{i=1}^{N} y^i=0_K\right\}. \tag{2.0}$$

Clearly Y_F is finite. Let $U(y)$ represent the utility image of a point y in Y_F, i.e.

$$U(y)=\{u^1(y^1),\dots,u^N(y^N)\} \quad \text{for} \quad y=(y^1,\dots,y^N) \text{ in } Y_F.$$

Also let

$$Y_{PF}=\{y\in Y_F|y \text{ is Pareto optimal}\}^5$$

The utility image of the Pareto Frontier Y_{PF} is denoted by $U_{PF}\equiv U(Y_{PF})$.

The bidding process works in the following manner: agents begin at period zero at the point 0_K of their (net) trading sets. Each agent i chooses a point b^i in its upper contour set $G^i(0_K)$ according to its conditional bidding distribution $P^i(\cdot|0_K)$. This constitutes its bid for that period. If $b=\{(b^1,\dots,b^N)\}$ lies in Y_F, i.e. is feasible, then a reallocation of endowments takes place, so that the agents' new endowments are given by $(b^1+\omega^1,\dots,b^N+\omega^N)$. If the collection of bids is not feasible, then agents remain at their current endowments. In the next period, exactly the same procedure is repeated and the process continues. Thus, the mechanism defines a finite Markov chain $\{B_t\}$ on Y_F and H–R–R show (Theorem 4.2) that *it converges to a point in*

[5] $y\in Y_{PF}$ if and only if $y\in Y_F$ and $\nexists\, \bar{y}\in Y_F$ such that $u^i(\bar{y}^i)\geqq u^i(y^i)$ for all $i=1,\dots,N$ with strict inequality for at least one i.

U_{PF} in finite time with probability 1. Once having reached a point in U_{PF}, the process stays there forever.

2.1. On the rate of convergence of the B-process with indivisible goods

Formally, of course, $\{B_t\}$ is a sequence of random elements from some probability space (Γ, F, P) taking values in Y_F. However, it is completely characterized by its transition matrix, and the initial distribution assigning probability one to the state $B_0=(0_K,\dots,0_K)$. We can now write down the transition probability matrix X corresponding to the Markov chain $\{B_t\}$ on Y_F. Let $y=(y^1,\dots,y^N)$ be the current state and $z=(z^1,\dots,z^N)$ be another state in Y_F. The transition probabilities are

$$X(z|y)=0 \quad \text{if} \quad [U(z)-U(y)]\notin \mathbb{R}^N_+, \tag{2.1}$$

$$X(z|y)=\prod_{i=1}^{N} P^i(z^i|y^i) \quad \text{if} \quad z\neq y \quad \text{and} \quad [U(z)-U(y)]\in R^N_+, \tag{2.2}$$

$$X(y|y)=1-\sum_{\{z\in Y_F,\, z\neq y\}} X(z|y). \tag{2.3}$$

Note that by the assumption on support, the transition probabilities are strictly positive if and only if z is not Pareto inferior to y. In particular, for every y, there is a positive probability of staying at y, which is 1 only if y is Pareto optimal.

In general, the mapping $U\colon Y_F\to R^N$ need not be one-to-one. However, the stochastic process $\{U(B_t)\}$ (which we shall henceforth refer to as $\{U_t\}$) is a Markov chain, as indicated by the following lemma.

Lemma 2.1. The B-process induces a Markov chain on $U(Y_F)$ with transition matrix V given by:

$$V(u'|u)=\begin{cases}\sum_{y'\in A(u')} X(y'|y) & \textit{if } (u'-u)\in R^N_+\\ 0 & \textit{otherwise}\end{cases}, \tag{2.4}$$

where $A(u')\equiv U^{-1}(u')\equiv\{y'\,|\,U(y')=u'\}$ and y' is any element of $U^{-1}(u)$.

Proof. See appendix.

The above implies $V(u|u)$ is 1 if and only if u is Pareto optimal. Furthermore, by the assumption on the support, if u is not Pareto optimal, there is a strictly positive probability of moving to some Pareto optimal state. It follows from the standard theory of finite Markov chains [see Kemeny and Snell (1960)] that the set of Pareto states are ergodic (absorbing) states and all others are transient. Results concerning absorbing Markov chains can be applied here, to demonstrate exponential convergence. Using the special structure of the process, we indicate how to get a 'fairly good' upper bound for the rate parameter in terms of the transition probabilities.

Lemma 2.2 in the appendix allows us to write down the transition matrix (starting from a non-optimal state u) as an upper triangular matrix V_u. Let h be the total number of states, g the number of Pareto optimal states and $n=h-g$ the number of non-Pareto optimal states. We have

$$V_u=\begin{pmatrix} \pi_{11} & \pi_{12} & \cdots & \pi_{1n} & \\ 0 & \pi_{21} & \cdots & \pi_{2n} & [W]_{n\times g} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & \pi_{nn} & \\ & & [O]_{g\times n} & & [I]_{g\times g} \end{pmatrix}. \tag{2.5}$$

We note that in light of Lemma 2.2, the first row refers to the state u. Also, W is the matrix of absorption probabilities starting from a non-optimal state. It follows from the assumptions on the B-process that each row of W has at least one strictly positive element. Finally, we have:

Proposition 2.1. Let V_u be as given in (2.5). Define $\lambda_u=\max_{ij}\pi_{ij}<1$. For every $\varepsilon>0$ such that $\lambda_u+\varepsilon<1$, there exists $C_\varepsilon>0$ such that for all $t\geqq 1$:

$$\Pr(U_t \text{ is not Pareto optimal} \mid U_0=u)\leqq C_\varepsilon(\lambda_u+\varepsilon)^t.$$

Proof. See appendix.

The above proposition shows that convergence from any original position occurs at least geometricaly. For comparability, we shall refer to this in terms of the exponential function henceforth.

Corollary 2.1. There exist $\alpha>0$ and $C>0$ such that for any initial probability distribution over the states U_0, $\Pr(U_t$ is not Pareto optimal$)\leqq C\,\mathrm{e}^{-\alpha t}$.

Proof. For each u in the non-Pareto optimal set, define λ_u corresponding to the transition matrix V_u, where λ_u is defined as in Proposition 2.1 above. Let $\mu=\max_u\lambda_u<1$. Fix any $\varepsilon>0$ satisfying $\mu+\varepsilon<1$ and choose C sufficiently large so that the condition in Proposition 2.1 is satisfied for every u which is non-optimal. Finally, let $\alpha=-\log(\mu+\varepsilon)$. □

Example 2.1. We calculate a *lower bound* for α (Corollary 2.1 implies that a higher value of α is associated with a faster rate of convergence to the Pareto frontier) in an economy with N agents and two indivisible goods. The total endowment of the economy is (r, r), r being a positive integer. All agents $i=(1,\ldots,N)$ have the same utility function $u^i(x_1^i, x_2^i)=\min(x_1^i, x_2^i)$. The 'Edgeworth box' is the set

$$E=\left\{(x^i):(x_1^i, x_2^i)\in Z_+^2 \text{ and } \sum_i x^i=(r;r)\right\}.$$

An allocation (x^i) is an element of E. Each agent i has a bidding distribution $P^i(\cdot)$ on Z_+^2; assume that $P^i(x)>0$ for all $x\in Z_+^2$. Now denote by $m^i=\min_{x\in E} P^i(x)>0$ and write $m=\min_i m^i$. Then m is the smallest probability with which any point in E is bid by any agent. It is not difficult to see that for any non-optimal allocation in E there are at least N Pareto optimal allocations in every agent's upper contour set. Thus the probability of moving to a Pareto optimal allocation is at least Nm^N. Using our terminology above, this implies that $\mu\leqq 1-Nm^N$, and $\alpha=\log\mu\geqq Nm^N$ (approximately). Two conjectures can be made from this calculation: first, as the number of agents increases, the bound decreases, and as the agents bid more extravagantly (lower m), the rate of convergence decreases as well. The simulation results discussed in section 4 lend additional support to these conjectures.

3. The surplus reallocation process in an economy with both divisible and indivisible goods[6]

The bidding process was developed also by H–R–R for an economy consisting exclusively of divisible goods. Suppose that there are M divisible goods. In H–R–R, the sets Y^i are assumed to have non-empty interior in R^M, and the initial bidding distributions $P=(P^1,\ldots,P^N)$ are assumed to be absolutely continuous with respect to the Lebesgue measure. If agent i is currently at position x^i, then $P^i(G^i(x^i))$ is the probability mass associated with the upper contour set. A positive number $\delta>0$ is fixed from the outset. The agent chooses a bidding cube of *radius* δ in the following manner: the conditional bidding distribution (CBD) is

$$P^i(y^i|x^i)=\begin{cases}0 & \text{if } y^i\notin G^i(x^i)\\ P^i(y^i)/P^i(G^i(x^i)) & \text{if } y^i\in G^i(x^i)\end{cases}. \tag{3.0}$$

[6]This modified process is due to V. Bala.

Agent i chooses the center of its bidding cube according to its CBD, which will be an M-dimensional point in its trading set, say $r^i=(r_1^i,\dots,r_M^i)$. The bidding cube is the M-dimensional set $\beta^i(r^i;x^i)$ given by $\{(r_1^i-\delta,r_1^i+\delta)\times\cdots\times(r_M^i-\delta,r_M^i+\delta\}\cap G^i(x^i)$ and is the set of all allocations the agent is willing to accept. The set of acceptable allocations is defined as $\beta^*(r;x)$ where $r=(r^1,\dots,r^N)$, $x=(x^1,\dots,x^N)$ and equals $\{\sum_{i=1}^N\beta^i(r^i;x^i)\}\cap Y_F$. $\beta^*(r;x)$ is a [possibly empty] subset of the set of feasible allocations which do not make any agent worse off than x. If it is non-empty, the referee chooses a point at random from this set which constitutes the new allocation from which bids will be made. If it is empty, then agents stay at their current allocation and make another bid using the same CBDs as before.

The major problem with this process is that the messages sent by each agent can be quite difficult to describe and communicate. When the center of the agent's bidding cube is close to the current allocation, the agent will have to transmit details of the local preference structure to the referee, which can be quite cumbersome. In addition, the set $\beta^*(r,x)$ is also quite difficult to compute for the referee even in small economies (this issue came to light in our simulation experiments). The simulations (see the remarks in section 4) on the speed of convergence led to a modified B-process – which we have called the Surplus Reallocation (SR) Process – with agents making only a single (point) bid in each period, and where any 'surplus' after bidding is reallocated by an averaging process. This modification also works for the 'mixed' case where some commodities are divisible, others are not. Our main result is that *the SR process converges to a weakly Pareto optimal outcome with probability one.* In what follows, an allocation is *weakly Pareto optimal* if there is no other feasible allocation yielding a strictly higher utility level for *all* agents. Mitsui (1981, 1984) also uses the same criterion (and assumption ED.6 of H–R–R also implies weak Pareto optimality in our context).

3.1. A formal model of a mixed economy

As before, Z_+ is the set of all non-negative integers. For two vectors $x=(x_k)$, $y=(y_k)$ we write $x\geqq y$ if $x_k\geqq y_k$ for all $x; x>y$ if $x\geqq y$ and $x\neq y$. We consider an exchange economy with N agents (indexed by $i\in I=\{1,2,\dots,N\}$), M divisible goods and K indivisible goods. For a commodity vector x (in $R_+^M\times Z_+^K$) we write $x=(\bar{x},\dot{x})$ where $\bar{x}$ (resp. $\dot{x}$) is the M-vector of divisible (resp. K-vector of indivisible) goods. The *consumption set* of agent i, denoted by C^i, is simply $C^i=R_+^M\times Z_+^K$. (To be sure, Z_+ has the discrete topology and any product space is assigned the product topology). The *initial endowment* of agent i is denoted by $\omega^i=(\bar{\omega}^i,\dot{\omega}^i)$, and the *set* Y^i *of net trades* is (as before) $Y^i\equiv C^i-\{\omega^i\}$. In what follows, it is assumed that the aggregate endowment of every good is strictly positive. The *utility function* u^i

of agent i is assumed continuous and weakly monotonic on Y^i.[7] (As in section 2, we define u^i first on C^i and then redefine on Y^i.)

We now describe the *bidding distribution* of agent i. Write $H^i \equiv Z_+^K - \{\dot{\omega}^i\}$ and $F^i = R_+^M - \{\bar{\omega}^i\}$. The sets $\{H^i\}$ are countable, and we enumerate these as $\{\dot{x}_1^i, \dot{x}_2^i, \ldots, \dot{x}_r^i, \ldots\}$. Let $f^i(\bar{x}_r^i, \dot{x}_r^i)$ be a continuous, strictly positive subdensity function on F^i for each r. In other words, for each $i=1,\ldots,N$, $f^i(\bar{x}^i, \dot{x}_r^i)$ satisfies:

$$\int_{F^i} f^i(\cdot, \dot{x}_r^i)\,\mathrm{d}\bar{x}^i > 0 \quad \text{for each} \quad \dot{x}_r^i \quad \text{and} \quad \sum_{(\dot{x}_r^i \in H^i)} \int_{F^i} f^i(\cdot, \dot{x}_r^i)\,\mathrm{d}\bar{x}^i = 1. \tag{3.1}$$

We also write $H = \prod_{i=1}^N H^i$, and note that H is countable.

Thus, we describe an agent formally by its set Y^i of all net trades, its utility function u^i, and its bidding distribution f^i. Next, for *divisible* goods, let us introduce the notion of a *surplus reallocation function* $\tau: R^{MN} \to R^{MN}$ as

$$\tau(\bar{x}) \equiv (\bar{y}^i) \quad \text{with} \quad \bar{y}^i \equiv \bar{x}^i - N^{-1} \sum_{1 \leqq j \leqq N} \bar{x}^j, \qquad i=1,\ldots,N. \tag{3.2}$$

Observe that $\sum_i \bar{y}^i = 0_M$ and that if each of the M components of $\sum_j \bar{x}^j$ is non-positive, then $\bar{y}^i \geqq \bar{x}^i$ for all i.

Recall that a commodity vector x is partitioned as $x = (\bar{x}, \dot{x})$ according to the divisible and indivisible goods. Let

$$Y_C = \left\{ x = (x^1, \ldots, x^N) : x^i \in R^M \times Z^K,\ \sum_{i=1}^N \bar{x}^i = 0_M,\ \sum_{i=1}^N \dot{x}^i = 0_K \right\}.$$

Recall [from (2.0)] that Y_F is the set of all feasible net trades. Then, $Y_F = \prod_i Y^i \cap Y_C$ is compact. Let $y \in Y_F$. Writing $y = (y^1, \ldots, y^i, \ldots, y^N)$, we define

$$G^i(y^i) = \{x^i \in Y^i : u^i(x^i) \geqq u^i(y^i)\}, \tag{3.3}$$

$$G^{+i}(y^i) = \{x^i \in Y^i : u^i(x^i) > u^i(y^i)\}, \tag{3.4}$$

[7] $x^i, y^i \in Y^i$ with $x^i \geqq y^i$ implies $u^i(x^i) \geqq u^i(y^i)$.

$$G^+(y)=\prod_{i=1}^{N} G^{+i}(y^i), \tag{3.5}$$

$$Y_{WPF}=\{y\in Y_F: \nexists\, z\in Y_F \text{ such that } u^i(z^i)>u^i(y^i) \text{ for all } i\}. \tag{3.6}$$

In eq. (3.6) above, Y_{WPF} denotes the weak Pareto frontier for the exchange economy. Points in this set will be referred to as 'weakly Pareto optimal allocations'. For each $\varepsilon>0$, write

$$\hat{Y}_\varepsilon=\{x\in Y_F: \text{there is } y\in Y_{WPF} \text{ such that } u^i(x^i)>u^i(y^i)-\varepsilon \text{ for all } i\}, \tag{3.7}$$

$$\bar{Y}_\varepsilon=Y_F\setminus\hat{Y}_\varepsilon. \tag{3.8}$$

Note that for each $\varepsilon>0$, $\hat{Y}_\varepsilon$ is relatively open in Y_F and $\bar{Y}_\varepsilon$ is compact. $\hat{Y}_\varepsilon$ is interpreted as the set of all feasible allocations which are ε away from the Pareto frontier in utility terms for each agent. It is not difficult to show that $Y_{WPF}=\bigcap_{\varepsilon>0}\hat{Y}_\varepsilon$. Furthermore if y is not weakly Pareto optimal, then for some $\eta>0$, $y\in\bar{Y}_\varepsilon$ for all $\varepsilon<\eta$. The probability mass of the 'upper contour set' at y^i is

$$P^i(G^i(y^i))=\sum_{\dot{x}^i_r\in H^i}\int_{F^i} \mathbf{1}_{G^i(y^i)}(\bar{x}^i,\dot{x}^i_r)f^i(\bar{x}^i,\dot{x}^i_r)\,\mathrm{d}\bar{x}^i. \tag{3.9}$$

The conditional density at $x^i=(\bar{x}^i,\dot{x}^i_r)\in Y^i$ given $y^i\in Y^i$ is defined as

$$f^i(x^i|y^i)=\begin{cases}0 & \text{if } x^i\notin G^i(y^i)\\ f^i(\bar{x}^i,\dot{x}^i)/P^i(G^i(y^i)) & \text{otherwise}\end{cases}. \tag{3.10}$$

It is easy to see that the function defined in (3.10) generates a probability measure on Y^i for every $y^i\in Y^i$.

3.2. A modification of the bidding process: Surplus reallocation

The surplus reallocation process also defines a stochastic process $\{B_t\}$ on the space of feasible allocations Y_F. In what follows the essential modifications are indicated. Agents begin at the initial state $B_0\equiv 0_{(M+K)N}$ (zero net trade/initial endowments) and bid for a commodity vector according to the conditional distribution (i.e., the ith agent chooses $b^i=(\bar{b}^i,\dot{b}^i)$ according to $f^i(\cdot|0_{M+K})$. If $\sum_i b^i=0$, i.e., the net trades are feasible, then a reallocation of endowments takes place so that the agents' new endowments are $(b^i+\omega^i)$ and we write $B_1=(b^i)$ where B_1 is the (random) state in period one. If $0>\sum\bar{b}^i$ and $\sum\dot{b}^i=0$, i.e. the total bid for divisible goods is less than zero and the total bid for indivisible goods is equal to zero, then the surplus of divisible commodities is redistributed according to (3.2), i.e., $B_1 = (\tau^i(\bar{b}), \dot{b}^i)$. Finally, if neither of the two cases above hold, then $B_1 \equiv B_0$, i.e. no 'action' takes place. In all the three cases, the process goes into the next round with B_1 in place of B_0, and the story is repeated. Lemmas 3.1 to 3.3 are preliminary analytical results, whose statements may be found in the appendix. The primary lemma is Lemma 3.4, from which the main result follows easily.

Lemma 3.4. Given $\varepsilon > 0$ there is $\gamma > 0$ such that:

$$\mu(y) = \Pr(B_t \in \hat{Y}_\varepsilon \mid B_{t-1} = y) \geqq \gamma, \text{ for all } t, \text{ for all } y \in Y_F.$$

In particular γ is independent of y and t but depends upon ε.

Proof. See the appendix. □

We can restate the lemma as saying that there exists a function $\gamma(\varepsilon)$ such that

$$\inf_{y \in Y_F} \mu(y) \geqq \gamma(\varepsilon) > 0 \quad \text{for all} \quad \varepsilon > 0,$$

where γ is defined as above. Let $U(\cdot)$ be the utility image of either an allocation or a set of allocations depending upon the context. In particular U_{WPF} is the image of the Y_{WPF}, $\bar{U}_\varepsilon = U(\bar{Y}_\varepsilon)$, and $\{U_t\} = \{U(B_t)\}$ is the utility image of the stochastic process of allocations. Since the SR process yields a sequence of allocations that is utility monotone for almost every sample path, the $\{U_t\}$ sequence is non-decreasing. Furthermore, since $\{U_t\}$ has as its range the image of a compact set under a continuous map, there is a limit random variable $U^* < \infty$ a.s. such that $\{U_t\}$ converges to U^*.

Theorem 3.1. $\Pr(U^* \in U_{WPF}) = 1$.

Proof. Exactly as in H–R–R (Theorem 5.2). For completeness, a sketch is given. Fix $\varepsilon > 0$. From the lemma, $\Pr(U_1 \in \bar{U}_\varepsilon) \leqq 1 - \gamma(\varepsilon)$. Using the Markov nature of the process, $\Pr(U_t \in \bar{U}_\varepsilon) \leqq (1 - \gamma(\varepsilon))^t$ so that $\Pr(U^* \in \bar{U}_\varepsilon) = 0$. Let $A = \bigcup_{n \geqq 1} \bar{U}_{(1/n)}$. Note that $A^c = U_{WPF}$. We have $\Pr(U^* \in A) = 0$ or $\Pr(U^* \in U_{WPF}) = 1$. □

4. Some simulation results

Section 4.1 discusses the performance of the B-process and the SR process in Scarf's example, while section 4.2 summarizes findings from simulations of economies with indivisible goods. The last section (4.3) studies the question of incentive compatibility in the context of a simple example.

4.1. Scarf's example

There are three agents ($i=1,2,3$) and three divisible goods (called x, y, z). The utility functions are given by $u^1(x,y,z)=\min(x,y)$ $u^2(x,y,z)=\min(y,z)$ and $u^3(x,y,z)=\min(x,z)$. The initial endowments are $\omega^1=(1,0,0)$, $\omega^2=(0,1,0)$, $\omega^3=(0,0,1)$. Let $Z(p)$ be the excess demand vector at prices p; we specify the tatonnement dynamics as given by $\mathrm{d}p/\mathrm{d}t=Z(p(t))$. There is a unique equilibrium price vector (up to a scalar multiple) at the prices $p^*=(1,1,1)$ which is globally unstable, in the sense that if the process starts at any disequilibrium price, it cycles forever without reaching p^*.

To define the Pareto frontier, we introduce a surplus function ζ. Given a feasible allocation (a^1,a^2,a^3) (feasibility means $\sum a^i=(1,1,1)$) define $\zeta: R^9 \to R^3$ as

$$\zeta(a^1,a^2,a^3)=[1-u^1-u^3,\ 1-u^1-u^2,\ 1-u^2-u^3]. \tag{4.0}$$

One can show that an allocation (a^1,a^2,a^3) is weakly Pareto optimal if and only if at most one element of $\zeta(a^1,a^2,a^3)$ is non-zero.[8]

In general, the Pareto frontier cannot be reached in finite time when there are divisible commodities. Consequently, we measure performance by fixing the total number of rounds at 1,000, and calculate the distance between the allocation attained by this point and the Pareto frontier. A natural measure of distance in this economy is given by the surplus function. We set $\rho(a)$ to be the minimum of the three components of $\zeta(a)$ for any feasible allocation a, where ζ has been defined by eq. (4.0). Then, $\rho(\omega)=1$, where ω is the initial endowment, and $\rho(x)=0$ for any weakly Pareto optimal allocation x.

A B-process simulation for the Scarf economy,[9] where the radius of the bid cube was 0.1, yielded allocations with $\rho=0.01$ (sample average over 100 realizations). Fig. 1, titled 'B-process simulation of Scarf economy' shows the surplus functions for a typical sample realization.

The surplus reallocation process (Section 3.2) was also simulated for the Scarf economy (see footnote 9). One hundred sample paths were simulated, and the process moved to within $\rho=0.11$ of the Pareto frontier on average within the first 1,000 bidding rounds. The surplus functions for a single

[8]This follows once we note that $\zeta(\cdot)=[(x^1-u^1)+(x^3-u^3)+x^2,\ (y^1-u^1)+(y^2-u^2)+y^3,\ (z^2-u^2)+(z^3-u^3)+z^1]$, where $a^i=(x^i,y^i,z^i)$, $i=1,2,3$. Now, $x^1 \geqq u^1$, $x^3 \geqq u^3$, $x^2 \geqq 0$, so that the first component (and similarly the others) is always non-negative. The claim follows from a careful case-by-case study using the above.

[9]The bidding distributions used in the simulations are as follows: for each agent, it is the product of independent exponential distributions, with parameter $\lambda=4$, one for each commodity. (An exponential distribution has density function given by $\mathbf{1}_{\{x \geqq 0\}} \lambda e^{(-\lambda x)}$].

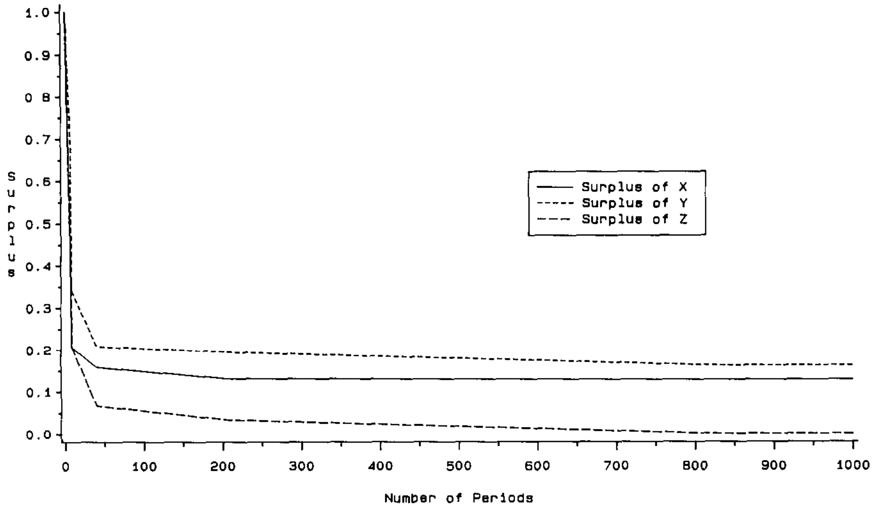

Fig. 1. B-process simulation of Scarf economy (surplus functions for one sample path).
Bid radius: 0.1 Total number of periods: 1,000.

sample path are shown in fig. 2, titled 'Surplus reallocation process for Scarf economy'.

The graphs reveal that most of the improvement occurs within the first 50 to 100 rounds. This is typical of most of the sample realizations, and suggests that the B-process or the SR process can be stopped after just a few periods and still realize most of the gains from trade.

4.2. Economies with indivisible goods

We consider economies with two goods and N agents each with a (common) utility function $u(x_1, x_2) \equiv \min(x_1, x_2)$. Even with this simplification, the dynamics of the B-process cannot be easily captured in qualitative terms. Simulation enables us to formulate conjectures on a scientific basis and to obtain counter-examples. Details of the computer programs are available in Bala (1988). Some related simulation results are given in Bala (1989).

We assume that agents choose their bidding distributions from the family of Discrete Normal distributions $\{DN(\mu, \sigma)\}$ where $\mu \geqq 0$, $\sigma > 0$ are parameters of the distribution. A random variable X is distributed as $DN(\mu, \sigma)$ if

$$\Pr(X = k) = \Pr(k < |\mathrm{N}(\mu, \sigma)| < k+1) \quad \text{for } k = 0, 1, 2, \ldots$$

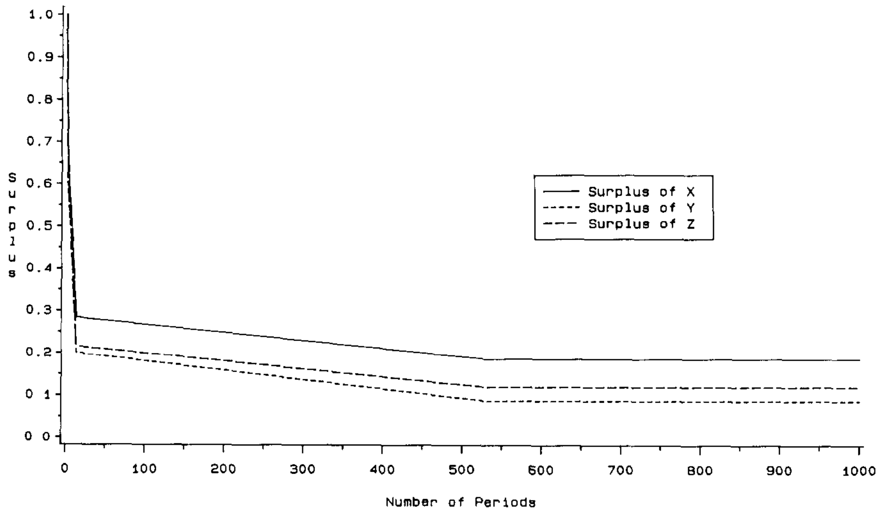

Fig. 2. Surplus reallocation process for Scarf economy (surplus functions for one sample path). Number of periods: 1,000.

Thus, this is a discrete analogue to the Normal distribution in which agents can independently control the 'mean' bid as its 'spread' by choosing appropriate values of μ and σ parameters.

It transpires that the following modification increases the rate of convergence: if total bids are less than aggregate endowments in the economy, then the surplus is redistributed at random among the agents. By 'random' it is meant that, when there are N agents in the economy and a surplus of p items of good X, then the probability that agent i will get x items of the surplus is $C(p,x)(1/N)^x(1-1/N)^{p-x}$.[10] It is easy to show that, for general exchange economies:

Proposition 4.1. The modification of the B-process proposed above will not affect the Markov structure of the process.

Proof. Omitted.

We are interested in obtaining an idea of how the speed of convergence varies with the searching process used by the agents. To this end, we simulate an economy with four agents and two indivisible goods. Agents 1

[10] $C(p,x)=p!/x!(p-x)!$

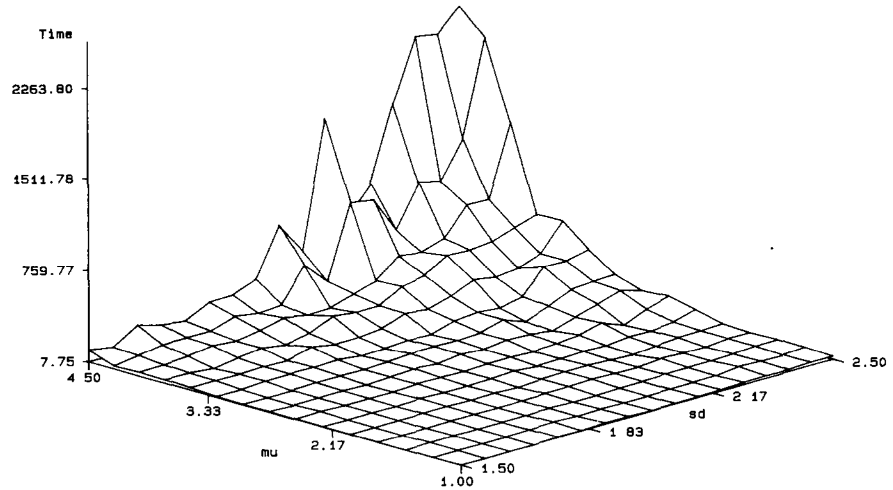

Fig. 3. Convergence time by parameter choice (four identical agents).
Agents have $u(x, y) = \min(x, y)$ and use $DN(mu, sd) * DN(mu, sd)$ bid distributions.
Replications: 20. Average standard error: $0.21 * (\text{mean})$.

and 3 each have endowments (0, 10) of the two goods, while agents 2 and 4 have endowments (10, 0) each. The (identical) bidding distribution of each agent is given of the product of independent $DN(\mu, \sigma) \times DN(\mu, \sigma)$ distributions. Fig. 3, titled 'Convergence time by parameter choice' summarizes the exercise. As can be seen, high μ and σ parameters are positively linked to the time taken to converge. Intuitively, a large μ implies, on average, that bids ask for much more than the total endowment. As a consequence, such bids are rejected as infeasible, increasing the convergence time. As σ rises, agents are experimenting more and more in their search for improving allocations, which perhaps makes it less likely that their bids will be compatible.

Some other features of the simulation exercises are noteworthy. First, there can be long periods with no trades even though Pareto improving trades exist, because agents use inappropriate search parameters relative to the economic environment. For example, a two agent economy, with each agent using parameters $\mu = 15$, $\sigma = 1$, in an environment with total endowments (5, 5) was simulated; the process was terminated after 99,999 rounds with no new trades. Second, the modification with 'random reallocation' can lead to substantial improvements in the speed of convergence. The intuition underlying this may be as follows: if agents are using 'inappropriate' search rules which underestimate the aggregate endowments, then there is a high probability that their bids will sum up to *less* than the total endowments. In the unmodified process, this implies that bids are rejected for *not* exhausting the total endowment, thus pushing up the convergence time. By contrast, the modification, which provides some justification for the process presented in section 3 above, accepts such bids, improving the dynamic performance of the process in these environments.

4.3. *Incentive compatibility*

The behavior assumption underlying the Walrasian model (each agent treating prices as given parameters) has been shown to be implausible (or incentive-incompatible) in the framework of 'small' economies [see, e.g., Hurwicz (1972, section 3)]. The B-process also faces difficulties in this direction. The next experiment throws light on this issue: each agent has an incentive to select an appropriate bidding distribution that will maximize the probability of the process getting absorbed into a Pareto optimal state most favorable to him (her). However, if both the agents pursue this objective, the expected time to converge to a Pareto optimal state tends to infinity.

Example 4.1. As before, let there be two agents with consumption sets $C^1 = C^2 = Z^2_+$ and utility functions $u^i(x^i_1, x^i_2) = \min(x^i_1, x^i_2)$ on $\mathbb{Z}^2_+$. Suppose that the total endowment in the economy is (2, 2) with agent 1 having both units of the first good and agent 2 having both units of the second. The Pareto frontier in the utility space is given by $\{(2,0),(1,1),(0,2)\}$. Assume that the bidding distribution on C^i factors out as the product of independent geometric distributions.[11] We shall assume that $\Pr\{\text{bid of } (x_1, x_2) | (0,0)\} = (1-p)^2 p^{x_1+x_2}$ for agent 1 and correspondingly for agent 2 with a parameter q instead of p. From the theory of Markov chains [see Iosifescu (1980)] the expected time to absorption and the probability distribution over absorbing states can be calculated. The implication seems to be that there is a trade-off between individual greed (i.e. the relative position on the frontier) and the expected amount of time taken to reach the frontier. Figure 4, titled 'Probability of absorption' depicts the probabilities of being absorbed into the Pareto optimal state (0, 2) as a function of the agents' parameter choices. From this graph it may be seen that agent 2 has an incentive to choose as high a parameter value q as possible, since this will maximize the probability of getting absorbed into the state most favorable to itself. The same argument applies to agent 1, for reasons of symmetry. However, as both

[11] A random variable X has a geometric distribution with parameter p (denoted $\mathrm{Ge}(p)$) if $\Pr(X=k) = (1-p)p^k$, for $k = 0, 1, 2, \ldots$ The corresponding conditional distribution is $\Pr(X = k | X \geq j) = (1-p)p^{k-j}$ for $k \geq j$.

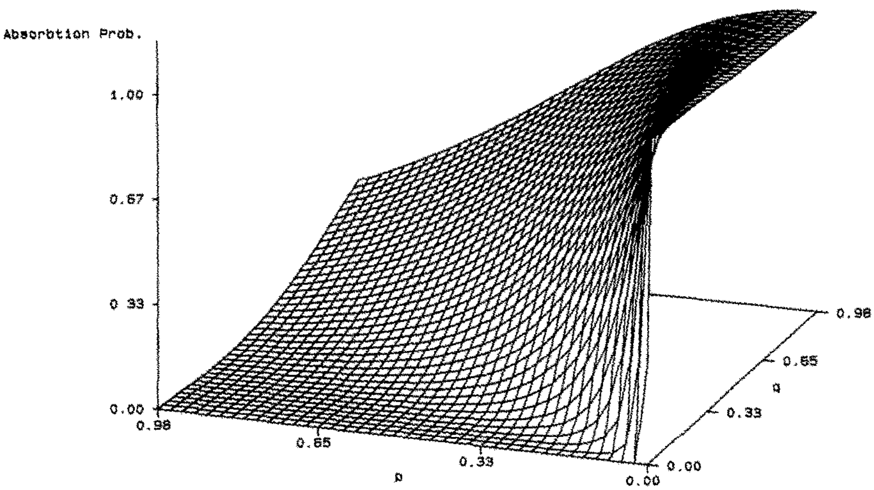

Fig. 4. Probability of absorption.
Absorption from utility state (0, 0) to state (0, 2).

agents increase their parameters, these cancel each other out [the probability of getting absorbed into the state (1, 1) increases] and the expected absorption time tends to infinity, as depicted in fig. 5, titled 'Expected absorption time'.

Appendix

Proof of Lemma 2.1

The proof uses a result from Kemeny and Snell (1960). We need the following definition:

Definition A.1. A Markov chain $\{B_t\}$ defined on a state space $Y_F = \{x, y, z, \ldots\}$ with transition matrix $X(z|y)$ is lumpable with respect to a partition $\{A_1, \ldots, A_h\}$ of Y_F if the lumped process is a Markov chain with transition matrix V, i.e. $\Pr(B_t \in A_k | B_{t-1} \in A_j, B_{t-2} \in A_m, \ldots) = V(k|j)$ for every A_j and A_k.

Let $p(k|y) = \sum_{z \in A_k} X(z|y)$ for state $y \in A_j$ in the original chain. Theorem 6.3.2 in Kemeny and Snell (p. 124) summarizes what we need know about lumpability

A necessary and sufficient condition for a Markov chain to be lumpable is that for every pair A_j and A_k, $p(k|y)$ has the same value for every $y \in A_j$.

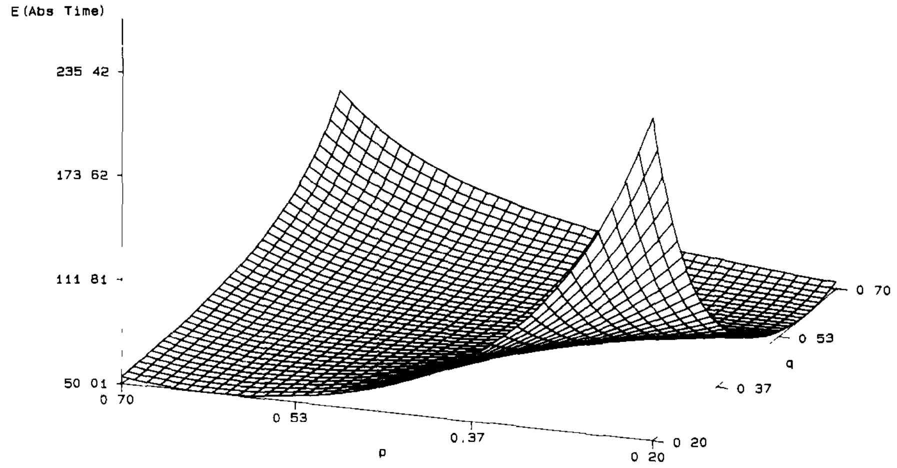

Fig. 5. Expected absorption time.
Absorption from utility state (0, 0) to Pareto frontier.

The transition matrix for the lumped chain is then given by $V(k|j) \equiv p(k|y)$ for some $y \in A_j$.

Note that we can choose $y \in A_j$ arbitrarily, as the assumption of lumpability implies that $p(k|y)$ is independent of the actual $y \in A_j$.

This result applies immediately to the B-process. If $\{B_t\}$ is the Markov chain in the set of feasible net trades Y_F and $\{U(B_t)\}$ is the corresponding process in utility space, let $\{u_1, \dots, u_h\}$ be the set of utility states. This partitions Y_F as $\{U^{-1}(u_j)\}_{1 \leq j \leq h}$ which we denote as $\{A_1, \dots, A_h\}$.

Let $y \in A_j$ and $z \in A_k$. Of course, $y = \{y^i\}_{i=1}^N$ and $z = \{z^i\}_{i=1}^N$. Now, if $u^i(y^i) > u^i(z^i)$ for some i, then clearly $p(k|y) = 0$ for every $y \in A_j$. Otherwise,

$$p(k|y) = \sum_{z \in A_k} X(z|y),$$

$$= \sum_{z \in A_k} \left[\prod_{i=1}^N P^i(z^i)/P^i(G^i(y^i)) \right],$$

$$= \left\{ \prod_{i=1}^N P^i(G^i(y^i)) \right\}^{-1} \sum_{z \in A_k} \prod_{i=1}^N P^i(z^i),$$

which is the same for all $y \in A_j$. The reason is as follows: if $U(x) = U(y)$, i.e. $x, y \in A_j$, then $P^i(G^i(x^i)) = P^i(G^i(y^i))$ for each agent i, since the upper contour sets $G^i(x^i)$ and $G^i(y^i)$ are the same. Hence, the sequence $\{U(B_t)\}$ is also a Markov chain with the transition probabilities given by the lumped chain. □

Starting from a point $u = \{u^i\}_{i=1}^N$ in the utility space $U(Y_F)$, let $S(u)$ be the set of all points $v = \{v^i\}_{i=1}^N$ in $U(Y_F)$ satisfying $v^i \geqq u^i$ for all i. Clearly, $u \in S(u)$. Define the map $\Psi_u : S(u) \to 2^{S(n)}$ as follows:

$$\Psi_u(v) = S(v) \quad \text{for} \quad v \in S(u).$$

It is to be noted that the map is well defined, i.e. $S(v) \subset S(u)$. In the language of graph theory, $\Psi_u(v)$ is the set of successors of v, including v as well. A *path* is an ordered subset $\{u_1, \ldots, u_r\}$ of $S(u)$ such that $u_1 = v$, $u_r = w$, and for all $1 \leqq k \leqq r-1$, $u_{k+1} \in \Psi_u(u_k)$, $u_{k+1} \neq u_k$. The set of all paths from v to w is denoted $\Theta(v, w)$. The B-process beginning at u induces the map Ψ_u with the following properties:

(i) $\Psi_u(v) = \{v\}$ if and only if v is Pareto optimal.
(ii) if $v \neq w$, then at least one of $\Theta(v, w)$ and $\Theta(w, v)$ is empty.

Lemma 2.2. Let h be the cardinality of $S(u)$. Then, there exists an ordering relation $<_a$ of $S(u)$ denoted $u_1 <_a u_2 <_a \cdots <_a u_h$ such that $u_1 = u$, and the transition probability $V(u_m | u_n) > 0$ only if $u_n <_a u_m$ or $u_m = u_n$. Furthermore, if g is the number of Pareto optimal states, then without loss of generality, the elements u_{h-g+1} to u_h can be taken to be the Pareto optimal states.

Proof. Define $S_k(u)$ as the set containing the first k ordered elements of $S(u)$, with $S_1(u) = \{u_1\} = \{u\}$. Let $S'_k(u)$ be $S(u) \cap [S_k(u)]^c$. This set consists of those elements which have not been ordered yet. We construct $S_{k+1}(u)$ inductively from $S_k(u)$ as follows: $S_{k+1}(u) = S_k(u) \cup \{v\}$, where v satisfies the following conditions:

(a) $v \in S'_k(u)$, i.e. it is chosen from the elements as yet unordered.
(b) There does not exist $w \in S'_k(u)$ different from v such that $v \in \Psi_u(w)$. In words, v is not a successor to another as yet unordered element w.
(c) If v and v' satisfy the above conditions and only one (say v') is Pareto optimal, then $u_{k+1} = v$. Otherwise, ties are broken arbitrarily.

We need to show that a v exists. Suppose not. Then (b) must be false. Then, for every $v \in S'_k(u)$ there exists $w \in S'_k(u)$ such that $v \neq w$ and $v \in \Psi_u(w)$. Since $S'_k(u)$ is a finite set, and every element has a predecessor other than itself, we can construct a cycle, i.e. property (ii) of Ψ_u is violated. This contradiction shows that the construction is not vacuous. Condition (c) above ensures that in ordering, all the Pareto optimal states come at end. □

Lemma 2.2 allows us to write down the transition matrix (starting from the state u) as an upper triangular matrix V_u. Recall that h is the total number of states, g the number of Pareto states and $n=h-g$ the number of non-Pareto optimal states. Also,

$$V_u=\begin{pmatrix} \pi_{11} & \pi_{12} & \dots & \pi_{1n} & \\ 0 & \pi_{21} & \dots & \pi_{2n} & [W]_{n\times g} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \dots & \pi_{nn} & \\ & & [O]_{g\times n} & & [I]_{g\times g} \end{pmatrix}.$$

We note that in light of the Lemma 2.2, the first row refers to the state u. Recall that each row of W has at least one strictly positive element.

For the convergence result (Proposition 2.1), we need the following additional lemma:

Lemma 2.3. Let $C(n,r)$ denote the combinatorial product (see footnote 10). Then

$$C(n,r)=\sum_{1\leqq s\leqq n-r+1} C(n-s,r-1).$$

Furthermore, for all $m,j\geqq 1$, we have:

$$C(j+m-1,m)=\sum_{1\leqq k\leqq j} C(k+m-2,m-1).$$

Proof. See Feller (1950, vol. 1, II.12). □

Proof of Proposition 2.1

Recall that we have defined λ_u to be $\max_{ij}\pi_{ij}$. The sum of the first row of the iterates V_u^t up to the nth column tell us the probability of remaining within the non-Pareto optimal set in the first t periods. Since each π_{ij} is bounded above by λ_u, it can be shown by iteration that:

$$\pi_{1j}^t\leqq C(j+t-2,t-1)\lambda_u^t, \tag{A.1}$$

where π_{1j}^t is the transition probability from 1 to j in t periods. Note that for fixed j, the above combinatorial product, $C(j+t-2,t-1)=(j+t-2)!/(t-1)!(j-1)!$ is a polynomial of degree $j-1$ in t. The result (1) is formally shown using induction. For $t=1$, the coefficient is $C(j-1,0)$, which equals 1. Assume (1) holds for $t=m$. Writing π_{1j}^{m+1} as:

$$\pi_{1j}^{m+1}=\sum_{1\leqq k\leqq h}\pi_{1k}^{m}\pi_{kj}=\sum_{1\leqq k\leqq j}\pi_{1k}^{m}\pi_{kj}\quad\text{since}\quad\pi_{kj}=0\text{ for }k>j$$

the last expression is dominated by $\lambda_u^{m+1}\sum_{1\leqq k\leqq j}C(k+m-2,m-1)$, and we employ Lemma 2.3 to continue the induction. Next,

$$\Pr(U_{t+1}\text{ is not Pareto optimal}|U_0=u)=\sum_{j\leqq n}\pi_{1j}^{t}\leqq P_{n-1}(t)\lambda_u^t$$

with $P_{n-1}(t)$ denoting a polynomial of degree $n-1$ in t. Since $t\geqq 1$, $P_{n-1}(t)$ is dominated by the function At^{n-1} for some $A>0$. If we can find C_ε such that $At^{n-1}\lambda_u^t\leqq C_\varepsilon(\lambda_u+\varepsilon)^t$ then we are done. This simplifies to requiring:

$$\log A+(n-1)\log t-\beta t\leqq\log C_\varepsilon,\quad\text{where }\beta=\log(1+\varepsilon/\lambda_u)$$

and since the function on the left has a global maximum at $t=(n-1)/\beta$, the proof of Proposition 2.1 is complete. □

We now take up the proofs of section 3. Before spelling out the proof of the main lemma, let us note:

Lemma 3.1. Let $a>0$. If (a_n) is a sequence of strictly positive reals,

$$\liminf_n (1/a_n)\geqq 1/a$$

if and only if $\limsup_n a_n\leqq a$.

Proof. Assume that $\limsup_n a_n\leqq a$. Given $\delta>0$, we need to show that there exists $N(\delta)$ such that

$$n\geqq N(\delta)\Rightarrow 1/a_n\geqq 1/a-\delta.$$

If $\delta\geqq 1/a$ then this is trivial. For $0<\delta<1/a$, choose $\varepsilon=\delta a^2/(1-\delta a)$. By assumption, there exists $N(\varepsilon)$ such that

$$n\geqq N(\varepsilon)\Rightarrow a_n\leqq a+\varepsilon.$$

Substituting for ε and rearranging, we get $1/a_n\geqq 1/a-\delta$ for $n\geqq N(\varepsilon)$.

The converse can be proved by similar means. □

Lemma 3.2. *Consider a finite collection* (b_n^i) *of sequences of non-negative reals* $(i=1,2,\dots,N)$. *Let* b^i, $i=1,\dots,N$ *be non-negative real numbers satisfying*

$$\liminf_n b_n^i \geqq b^i, \qquad i=1,\dots,N.$$

Then,

$$\liminf_n \prod_{i=1}^N b_n^i \geqq \prod_{i=1}^N b^i.$$

Proof. If $b^i=0$ for some i, then the result is trivial. Assume therefore that $b^i>0$ for all i. By assumption, for any δ satisfying $0<\delta<\min_i b^i$, there exists $N(\delta)$ such that if $n \geqq N(\delta)$ then $b_n^i \geqq b^i-\delta$ for all i. Hence, for $n \geqq N(\delta)$,

$$\prod_{i=1}^N b_n^i \geqq \prod_{i=1}^N (b^i-\delta) = \prod_{i=1}^N b^i - \phi(\delta),$$

where $\phi(\delta)$ is a polynomial in δ satisfying $\phi(\delta)\to 0$ as $\delta \to 0$. The result follows easily. □

Lemma 3.3. *If* $\{y_n^i\}\to y^i$ *in* Y^i, *and* $(\bar{x}^i;\dot{x}^i)\in G^{+i}(y^i)$, *then*

$$\liminf_n f^i((\bar{x}^i;x^i)\,|\,y_n^i) \geqq f^i((\bar{x}^i;x_i)\,|\,y^i). \tag{A.2}$$

Proof. For convenience, the 'i' superscripts will be omitted. Recall that $f((\bar{x};\dot{x})\,|\,y)=f((\bar{x};\dot{x}))/P(G(y))$, where $P(G(y))$ is the probability mass in the upper contour set, i.e.

$$P(G(y)) = \sum_{x\in H^i} \int_F \mathbf{1}_{G(y)}(\bar{x},\dot{x}) f(\bar{x};\dot{x})\,\mathrm{d}\bar{x}.$$

Now, using Lemma 3.1, the validity of (2) is equivalent to showing that

$$\limsup_n P(G(y_n)) \leqq P(G(y)).$$

There are two cases to consider: (i) $u(y_n)\geqq u(y)$ and (ii) $u(y_n)<u(y)$, for all n. In case (i) $G(y_n)\subset G(y)$ so that the inequality follows from the monotonicity of the integral. In case (ii), $G(y)=\bigcap_{n\geqq 1} G(y_n)$, where the sequence of sets can be taken to be decreasing. Then, from the continuity property of measures, we have $\lim P(G(y_n))=P(G(y))$, and the result follows. □

Proof of Lemma 3.4. Recall that $\mu(y)=P(B_t\in \hat{Y}_\varepsilon | B_{t-1}=y)$. If $y\in \hat{Y}_\varepsilon$ then by monotonicity of preferences and the fact that agents bid in $G^i(y^i)$ with probability 1, we get $\mu(y)=1$. We can thus restrict attention to $y\in \bar{Y}_\varepsilon$.

We begin by defining the set Δ as

$$\Delta=\{x=(x^1,\dots,x^N)\,|\,\textstyle\sum \bar{x}^i\leqq 0, \sum \dot{x}^i=0\}\cap \prod_{i=1}^{N} Y^i.$$

Δ is the set of all possible bids by agents which are globally feasible. Note that for divisible commodities, this includes bids that add up to less than the total endowments. We will then define $\{\tau^i(\bar{x});\dot{x}^i\}$ on Δ to be the allocation obtained from $x\in\Delta$ by redistributing the surplus (if any) of divisible goods.

Suppose $y\in \bar{Y}_\varepsilon$. Here, $y=\langle y^1,\dots,y^N\rangle$ where $y^i\in Y^i$. Define

$$A(\dot{x};y)=\{x\in\Delta\,|\,\text{for all } i, x^i\in G^{+i}(y^i), \{\tau^i(\bar{x});\dot{x}^i\}\in \hat{Y}_\varepsilon\}.$$

In words, $A(\dot{x};y)$ is the set of all bids that are:

(1) Strongly individually rational (i.e. yielding strictly higher utility) given the current allocation y^i and a fixed vector of indivisible goods $\dot{x}^i$ for each i.
(2) Globally feasible given the total endowment.
(3) After distributing the surplus of divisible goods, if any, the position in Y_F is ε-close to the frontier.

It may be noted that $A(\dot{x};y)$ will be empty for all but a finite number of $\dot{x}$, since the total endowments of goods of indivisible goods is finite. For $\dot{x}\in H$, let

$$\xi(\dot{x};y)=\int_{\bar{x}\in R^{MN}} \mathbf{1}_{A(\dot{x};y)}(\bar{x};\dot{x})\prod_{i=1}^{N} f^i((\bar{x}^i;\dot{x}^i)\,|\,y^i)\mathrm{d}\bar{x}.$$

Fix $\dot{x}\in H$ and denote by $h(\bar{x})$ the function inside the integral. Of course, $\xi(\dot{x},y)=\int h(\bar{x})\,\mathrm{d}\bar{x}$ where the argument y is suppressed. We show that $\xi(\dot{x};y)$ is lower semi-continuous on $\bar{Y}_\varepsilon$. To this end, let $y_n\to y$ in $\bar{Y}_\varepsilon$. We have:

$$\xi(\dot{x};y_n)=\int_{\bar{x}\in R^{MN}} \mathbf{1}_{A(\dot{x};y)}(\bar{x};\dot{x})\prod_{i=1}^{N} f^i((\bar{x};\dot{x}^i)\,|\,y_n^i)\,\mathrm{d}\bar{x} \tag{A.3}$$

and denoting the inside term by $g_n(\bar{x})$, we use Fatou's Lemma (the functions are non-negative) to obtain:

$$\liminf_n \xi(\dot{x};y_n)=\liminf_n \int g_n\,\mathrm{d}\bar{x}\geqq \int \liminf_n g_n\,\mathrm{d}\bar{x}. \tag{A.4}$$

Now, $\liminf_n g_n(\bar{x}) \geqq h(\bar{x})$, which we prove below. To show that:

$$\liminf_n g_n(\bar{x}) \equiv \liminf_n \mathbf{1}_{A(x;y_n)}(\bar{x}, \dot{x}) \prod_{i=1}^{N} f^i((\bar{x}^i; \dot{x}^i) | y_n^i)$$

$$\geqq \mathbf{1}_{A(\dot{x},y)}(\bar{x}; \dot{x}) \prod_{i=1}^{N} f^i((\bar{x}^i; \dot{x}^i) | y^i) \equiv h(\bar{x}) \tag{A.5}$$

we put

$$\mathbf{1}_{A(x;y_n)}(\bar{x}; \dot{x}) = a_n \quad \text{and} \quad \mathbf{1}_{A(x;y)}(\bar{x}; \dot{x}) = a,$$

and also

$$\prod_{i=1}^{N} f^i((\bar{x}^i; \dot{x}^i) | y_n^i) = b_n \quad \text{and} \quad \prod_{i=1}^{N} f^i((\bar{x}^i; \dot{x}^i) | y^i) = b,$$

for fixed $\dot{x}$. Then, if we can show that $\liminf_n a_n \geqq a$, and $\liminf_n b_n \geqq b$, Lemma 3.2 tells us that $\liminf_n a_n b_n \geqq ab$, which is what we require. To show that $\liminf_n a_n \geqq a$, i.e. that $\liminf_n \mathbf{1}_{A(\dot{x};y_n)}(\bar{x}; \dot{x}) \geqq \mathbf{1}_{A(\dot{x};y)}(\bar{x}; \dot{x})$ we need only look at those x such that $(\bar{x}; \dot{x}) \in A(\dot{x}; y)$. By the definition of $A(\cdot, \cdot)$, if the above does not hold, there is a subsequence $\{n_k\}$ such that for some agent i, we have

$$u^i(y_{n_k}^i) \geqq u^i(\bar{x}^i; \dot{x}^i) > u^i(y^i).$$

Since $y_{n_k}^i \to y^i$, continuity yields a contradiction. Now, Lemma 3.3 shows that for each i,

$$\liminf_n f^i((\bar{x}^i; \dot{x}^i) | y_n^i) \geqq f^i((\bar{x}^i; \dot{x}^i) | y^i). \tag{A.6}$$

Applying Lemma 3.2, we get

$$\liminf_n \prod_{i=1}^{N} f^i((\bar{x}^i; \dot{x}^i) | y_n^i) \geqq \prod_{i=1}^{N} f^i(\bar{x}^i; \dot{x}^i) | y^i).$$

Thus, we get $\liminf b_n \geqq b$, and we have proved (A.5), i.e. that $\liminf_n g_n \geqq h$. By monotonicity of the integral we then obtain:

$$\int \liminf_n g_n(\bar{x})\, d\bar{x} \geqq \int h(\bar{x})\, d\bar{x} \equiv \xi(\dot{x}; y).$$

Combining this with (A.4) above, we get $\liminf_n \xi(\dot{x}; y_n) \geqq \xi(\dot{x}; y)$.

Now, define $\tilde{\mu}(y)$ as $\tilde{\mu}(y)=\sum_{\dot{x}\in H}\xi(\dot{x};y)$.

Since the total endowments of indivisible goods is finite, the sum on the right-hand side of the above expression is actually the sum of a finite number of $\xi(\cdot)$ functions. Consequently, $\tilde{\mu}(\cdot)$ is also lower semi-continuous and as $\bar{Y}_\varepsilon$ is compact, $\tilde{\mu}(\cdot)$ attains its minimum at some $y^*\in\bar{Y}_\varepsilon$. We show now that $\tilde{\mu}(y^*)>0$. This may be done as follows:

Choose a point $x^*\in Y_{WPF}\cap G^+(y^*)$. Of course $x^*=(x^{1*},\ldots,x^{N*})$. As usual $\bar{x}^{i*}$ is the vector of divisible goods associated with x^* for agent i, with corresponding vector of indivisible goods being $\dot{x}^{i*}$. A small problem may arise because $\bar{x}^{i*}$ may be on the boundary of the trading set of agent i. Given that the aggregate endowment of every good is strictly positive, it is clear that for every divisible good k, there must be some agent j for whom $\bar{x}_k^{j*}$ (this denotes the net trade of the kth good by agent j) is greater than $-\omega_k^j$. We use continuity to pick $(\bar{x}^0;\dot{x}^*)$ in $\hat{Y}_\varepsilon\cap G^+(y^*)$ such that for each agent i, $\bar{x}^{i0}$ is in the interior of its set F^i. Now, since $E^i\equiv\{h^i|u^i(y^{i*})<u^i((h^i;\dot{x}^{i*}))<u(\bar{x}^{i0};\dot{x}^{i*})\}$ is an open set in F^i for all i, choose an open ball $N(\bar{t}^i)\subset E^i$ centered at a point $\bar{t}^i\in F^i$ and satisfying

(a) $\bar{x}^i\in N(\bar{t}^i)$ implies $\bar{x}^i<\bar{x}^{i0}$ (component-wise).
(b) $\bar{x}^i\in N(\bar{t}^i)$ implies $u^i((\bar{x}^i;\dot{x}^{i*}))>u^i(x^{i*})-\varepsilon$.

Since $\bar{x}^{0i}$ is in the interior of F^i, the above are possible due to continuity of the utility functions.

Now we note, using monotonicity, that if agents pick bids $\bar{x}^i$ in $N(\bar{t}^i)$ along with $\dot{x}^{i*}$ for each i, then there will be a surplus in each good, which after reallocation will lead to a position in $\hat{Y}_\varepsilon$. In other words,

$$W=\prod_{i=1}^{N}[N(\bar{t}^i)\times\{\dot{x}^{*i}\}]\subset A(\dot{x}^*;y^*).$$

But, since $N(t^i)$ is, for each i, an open set in $\text{proj}_{F^i}G^{+i}(y^{*i})$ they are bid with strictly positive probability so that $\tilde{\mu}(y^*)\geqq\Pr(W)>0$.

It is not difficult to see that $\mu(\cdot)\geqq\tilde{\mu}(\cdot)$ since the latter function is calculated assuming that agents bid in their strict upper contour sets whereas they actually bid in the weak upper contour sets.

This proves the result, since $\tilde{\mu}$ is strictly positive at its minimum value. □

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